Quaternions: Image Recognition

Introduction and Rotations

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Dyalog' 18
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Beyond $\mathbb{C}$

Real and complex fields and ?

- $\mathbb{R}$ is topologically complete.
  $x^2 + 1 = 0$ has no solution.
- $\mathbb{C}$ is algebraically closed: Every polynomial has a root. There are no "small" fields above $\mathbb{C}$
  If $K\big|_{\mathbb{C}} < \infty$ we get

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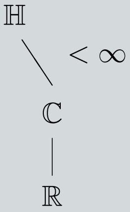

- $\mathbb{R}$ is topologically complete.
  $x^2 + 1 = 0$ has no solution.
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- Theorem of Gelfand-Mazur: Every finite dimensional skew field containing $\mathbb{R}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$: skew field of quaternions or Hamiltonians (William Rowam Hamilton, Irish mathematician and physicist, 1805 (Dublin) - 1865 (Dunsink near Dublin)).

Skew Field $\mathbb{H}$ as Complex Matrices

$\mathbb{H} \subseteq \mathbb{C}^{2,2}$

$\mathbb{C}^{2,2}$
|
$\mathbb{C}$
|
$\mathbb{R}$

$\mathbb{C}^{2,2}$ too big: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Skew Field $\mathbb{H}$ as Complex Matrices

$\mathbb{H} \subseteq \mathbb{C}^{2,2}$: Definition Quaternions / Hamiltonians

$\mathbb{C}^{2,2}$ — $\mathbb{C}$ — $\mathbb{R}$

$$h_0 = \mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad h_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

$$h_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

$$\begin{aligned} \mathbb{H} &:= \{a\,h_0 + b\,h_1 + c\,h_2 + d\,h_3 \mid a, b, c, d \in \mathbb{R}\} \\ &= \left\{ \begin{pmatrix} a + b\,i & c + d\,i \\ -c + d\,i & a - b\,i \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} v & w \\ -\overline{w} & \overline{v} \end{pmatrix} \middle| \; v, w \in \mathbb{C} \right\} \end{aligned}$$

Skew Field $\mathbb{H}$ as Complex Matrices

1. $\mathbb{H}$ is closed under matrix multiplication and addition. It contains the identity matrix and thus is a ring with identity.

$$\begin{pmatrix} a_1 + b_1\,\mathrm{i} & c_1 + d_1\,\mathrm{i} \\ -c_1 + d_1\,\mathrm{i} & a_1 - b_1\,\mathrm{i} \end{pmatrix} \cdot \begin{pmatrix} a_2 + b_2\,\mathrm{i} & c_2 + d_2\,\mathrm{i} \\ -c_2 + d_2\,\mathrm{i} & a_2 - b_2\,\mathrm{i} \end{pmatrix}$$

$$= \begin{pmatrix} a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2 + (a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2)\,\mathrm{i} & a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2 + (a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2)\,\mathrm{i} \\ -c_1 a_2 - d_1 b_2 - a_1 c_2 + b_1 d_2 + (-c_1 b_2 + d_1 a_2 + a_1 d_2 + b_1 c_2)\,\mathrm{i} & -c_1 c_2 - d_1 d_2 + a_1 a_2 - b_1 b_2 + (-c_1 d_2 + d_1 c_2 - a_1 b_2 - b_1 a_2)\,\mathrm{i} \end{pmatrix}$$

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   $\mathbb{H}$ contains three copies of the complex numbers.

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3. $h_1 h_2 = h_3,\ h_2 h_3 = h_1,\ h_3 h_1 = h_2$ und
   $h_2 h_1 = -h_3,\ h_3 h_2 = -h_1,\ h_1 h_3 = -h_2$.
   These rules are well known from the cross product on $\mathbb{R}^3$. Hence, this multiplication is not commutative.

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   $h_2 h_1 = -h_3,\ h_3 h_2 = -h_1,\ h_1 h_3 = -h_2$.
4. The map

$$\Phi : \left\{ \begin{array}{ccc} (\mathbb{R}^4, +) & \rightarrow & (\mathbb{H}, +) \\ (a, b, c, d) & \mapsto & \begin{pmatrix} a + b\,i & c + d\,i \\ -c + d\,i & a - b\,i \end{pmatrix} \end{array} \right\}$$

respects vector addition / matrix addition and scalar multiplication. So it is a vector space homomorphism.

Skew Field $\mathbb{H}$ as Complex Matrices

Theorem

$\mathbb{H}$ is a skew field (division ring) with centre $\mathbb{R}\,\mathrm{h}_0$.

Proof:

1. $$\begin{pmatrix} a+b\,\mathrm{i} & c+d\,\mathrm{i} \\ -c+d\,\mathrm{i} & a-b\,\mathrm{i} \end{pmatrix}^{-1} = \frac{1}{a^2+b^2+c^2+d^2}\begin{pmatrix} a-b\,\mathrm{i} & -c-d\,\mathrm{i} \\ c-d\,\mathrm{i} & a+b\,\mathrm{i} \end{pmatrix}$$
2. Direct calculations verify the centre.

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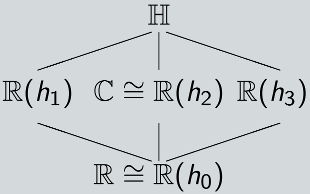
Skew Field $\mathbb{H}$ as Complex Matrices

Summary

1. $\left\{ \mathrm{h}_0 = \mathrm{Id},\ \mathrm{h}_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \mathrm{h}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathrm{h}_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \right\}$ *is a basis of* $\mathbb{H}$.
2. $\mathbb{H}$ *contains* $\mathbb{R}(h_i)$, $(i, \ldots, 3)$ *which are three copies of the complex numbers whose intersection is* $\mathbb{R} \cong \mathbb{R}(h_0)$, *the centre of* $\mathbb{H}$.



The Skew Field of Quaternions $(\mathbb{R}^4, +, \cdot)$

Remark

$(\mathbb{R}^4, +, \cdot)$ *with vector addition and the following multiplication*

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} \;\widehat{=}\; \begin{pmatrix} a_1 + b_1\,\mathrm{i} & c_1 + d_1\,\mathrm{i} \\ -c_1 + d_1\,\mathrm{i} & a_1 - b_1\,\mathrm{i} \end{pmatrix} \cdot \begin{pmatrix} a_2 + b_2\,\mathrm{i} & c_2 + d_2\,\mathrm{i} \\ -c_2 + d_2\,\mathrm{i} & a_2 - b_2\,\mathrm{i} \end{pmatrix}$$

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is a skew field isomorphic to $(\mathbb{H}, +, \cdot)$, *which is denoted by* $(\mathbb{H}, +, \cdot)$ *too. The inverse or reciprocal element is*

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}^{-1} = \frac{1}{a^2 + b^2 + c^2 + d^2} \begin{pmatrix} a \\ -b \\ -c \\ -d \end{pmatrix}$$

APL-Functions

Dyalog APL

```
r←a Hmul b
r←a[1]×b
r←r+a[2]×¯1 1 ¯1 1×b[2 1 4 3]
r←r+a[3]×¯1 1 1 ¯1×b[3 4 1 2]
r←r+a[4]×¯1 ¯1 1 1×⌽b

Hinv←{((1↑ω),-1↓ω)÷+/ω×ω}

Hdiv←{α Hmul Hinv ω}

Hcon←{(1↑ω),-1↓ω}

HsDi←{(α Hmul ω)-ω Hmul α}
```

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```

```
      Hinv 0 0 1 1
0 0 ¯0.5 ¯0.5
      0 1 0 0 Hmul 0 0 1 0
0 0 0 1
      0 1 0 0 HsDi 0 0 1 0
0 0 0 2
```

Complex Conjugate and Norm

Definition (Conjugate, Norm)

1. *Complex Conjugation* $^* : \mathbb{H} \to \mathbb{H}$ *is defined by*

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}^* = \begin{pmatrix} a \\ -b \\ -c \\ -d \end{pmatrix} \text{ or } \begin{pmatrix} a + b\,\mathrm{i} & c + d\,\mathrm{i} \\ -c + d\,\mathrm{i} & a - b\,\mathrm{i} \end{pmatrix}^* = \begin{pmatrix} a - b\,\mathrm{i} & -c - d\,\mathrm{i} \\ c - d\,\mathrm{i} & a + b\,\mathrm{i} \end{pmatrix}.$$

It is an additive automorphism and a multiplicative antiautomorphism on $\mathbb{H}$.

2. *The norm* $N : \mathbb{H} \to \mathbb{R}_{\geq 0}$ *of a quaternion is*

$$N\left(\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}\right) = a^2 + b^2 + c^2 + d^2 = \begin{vmatrix} a + b\,\mathrm{i} & c + d\,\mathrm{i} \\ -c + d\,\mathrm{i} & a - b\,\mathrm{i} \end{vmatrix}.$$

Unit Quaternions

Remark

For $q_1, q_2 \in \mathbb{H}$ we have $N(q_1 \cdot q_2) = N(q_1)N(q_2)$. So N is a homomorphism $(\mathbb{H}, \cdot)$ onto $(\mathbb{R}_{\geq 0}, \cdot)$.

Proof: $N(q_i) = det(q_i)$

Theorem

Für $S := N^{-1}\{1\} = \{q \in \mathbb{H} \mid N(s) = 1\}$ gilt $S \cong \mathrm{SU}(2, \mathbb{C})$. S is the set of all unit quaternions.

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Real and imaginary part

Definition

The real part of a quaternion $a\,\mathrm{h}_0 + b\,\mathrm{h}_1 + c\,\mathrm{h}_2 + d\,\mathrm{h}_3$ *is* a*, its imaginary part* $\begin{pmatrix} b \\ c \\ d \end{pmatrix}$.

In the decomposition
$\mathbb{H} = \mathrm{h}_0\,\mathbb{R} \oplus \mathrm{h}_1\,\mathbb{R} \oplus \mathrm{h}_2\,\mathbb{R} \oplus \mathrm{h}_3\,\mathbb{R} \cong \mathbb{R} \oplus \mathbb{R}^3 \cong \mathbb{R} \oplus V,\ V := \mathbb{R}^3$ *denotes the set of all imaginary parts.*

Real and imaginary part

Remark (Multiplikation)

Given $a,\ a_i \in \mathbb{R}$ *und* $\vec{v},\ \vec{v}_i \in V(i = 1, 2)$ *we have*

- $$\begin{pmatrix} a_1 \\ \vec{v}_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ \vec{v}_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 - \langle \vec{v}_1,\ \vec{v}_2 \rangle \\ a_1 \vec{v}_2 + a_2 \vec{v}_1 + \vec{v}_1 \times \vec{v}_2 \end{pmatrix}.$$

  Multiplication restricted to V corresponds to the cross product.

Real and imaginary part

Remark (Multiplikation, Inverse)

Given $a,\ a_i \in \mathbb{R}$ *und* $\vec{v},\ \vec{v}_i \in V(i = 1, 2)$ *we have*

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  Multiplication restricted to V *corresponds to the cross product.*

- $$\begin{pmatrix} a \\ \vec{v} \end{pmatrix}^{-1} = \frac{1}{a^2 + \|\vec{v}\|^2} \begin{pmatrix} a \\ -\vec{v} \end{pmatrix}$$

Real and imaginary part

Remark (Polar Representation of Unit Quaternions)

For $a,\ a_i \in \mathbb{R}$ *and* $\vec{v},\ \vec{v}_i \in V(i = 1, 2)$ *we get*

- $$S = \left\{ \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha)\,\hat{\omega} \end{pmatrix} \middle| \alpha \in [0, 2\pi) \wedge \hat{\omega} \in \{\vec{v} \in \mathbb{R}^3 \mid \|\vec{v}\| = 1\} \right\}$$

  This notation of a unit quaternion is called polar representation.

Real and imaginary part

Remark (Polar Representation of Unit Quaternions, Conjugation)

For $a,\ a_i \in \mathbb{R}$ *and* $\vec{v},\ \vec{v}_i \in V(i = 1, 2)$ *we get*

- *Conjugation with a unit quaternion* $\begin{pmatrix} \cos(\alpha) \\ \sin(\alpha)\,\hat{\omega} \end{pmatrix}$ *yields*

$$
\begin{aligned}
& \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha)\,\hat{\omega} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vec{v} \end{pmatrix} \cdot \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha)\,\hat{\omega} \end{pmatrix}^{-1} \\
= & \begin{pmatrix} 0 \\ (\cos^2(\alpha) - \sin^2(\alpha))\vec{v} + 2\,\langle \vec{\omega},\ \vec{v}\,\rangle\,\vec{\omega} + 2\cos(\alpha)\vec{\omega} \times \vec{v} \end{pmatrix} \\
= & \begin{pmatrix} 0 \\ \cos(2\alpha)\vec{v} + 2\sin^2(\alpha)\,\langle \hat{\omega},\ \vec{v}\,\rangle\,\hat{\omega} + \sin(2\alpha)\hat{\omega} \times \vec{v} \end{pmatrix}
\end{aligned}
$$

Real and imaginary part

Remark (Polar Representation of Unit Quaternions, Conjugation)

For $a,\ a_i \in \mathbb{R}$ *and* $\vec{v},\ \vec{v}_i \in V(i = 1, 2)$ *we get*

- *Conjugation with a unit quaternion* $\begin{pmatrix} \cos(\alpha) \\ \sin(\alpha)\hat{\omega} \end{pmatrix} = \begin{pmatrix} \omega_0 \\ \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$ *can be expressed by a rotational matrix*

$$D_{\omega,\alpha} = \begin{pmatrix} \omega_0^2 + \omega_x^2 - \omega_y^2 - \omega_z^2 & 2(\omega_x\omega_y - 2\omega_0\omega_z) & 2(\omega_0\omega_y + \omega_x\omega_z) \\ 2(\omega_0\omega_z + \omega_x\omega_y) & \omega_0^2 - \omega_x^2 + \omega_y^2 - \omega_z^2 & 2(\omega_y\omega_z - \omega_0\omega_x) \\ 2(\omega_x\omega_z - \omega_0\omega_y) & 2(\omega_0\omega_x + \omega_y\omega_z) & \omega_0^2 - \omega_x^2 - \omega_y^2 + \omega_z^2 \end{pmatrix}.$$

on V*.*

Rotations

Theorem

Conjugation with a unit quaternion $\begin{pmatrix} \cos(\alpha) \\ \sin(\alpha)\,\hat{\omega} \end{pmatrix}$ *yields a rotation around* $\hat{\omega}$ *with the angle* 2α.

```
      ,s←∊(2 1○○15÷180)×¨1 (0 0 1)
0.9659258263 0 0 0.2588190451

      Hdrmat s
0.8660254038 ¯0.5          0
0.5           0.8660254038 0
0             0            1

      s Hdreh 0,v←1 2 3
0 ¯0.1339745962 2.232050808 3
      (Hdrmat s)+.×v
¯0.1339745962 2.232050808 3
```

Rotations

Theorem

Conjugation with a unit quaternion $\begin{pmatrix} \cos(\alpha) \\ \sin(\alpha)\,\hat{\omega} \end{pmatrix}$ *yields a rotation around* $\hat{\omega}$ *with the angle* 2α*.*

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Rotations

Theorem

Conjugation with a unit quaternion $\begin{pmatrix} \cos(\alpha) \\ \sin(\alpha)\,\hat{\omega} \end{pmatrix}$ *yields a rotation around* $\hat{\omega}$ *with the angle* 2α.

Proof:

$$\begin{pmatrix} \cos(\alpha) \\ \sin(\alpha)\,\hat{\omega} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vec{v} \end{pmatrix} \cdot \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha)\,\hat{\omega} \end{pmatrix}^{-1} =$$

$$\begin{pmatrix} 0 \\ \cos(2\alpha)\vec{v} + 2\sin^2(\alpha)\langle \hat{\omega}, \vec{v} \rangle \hat{\omega} + \sin(2\alpha)\hat{\omega} \times \vec{v} \end{pmatrix}$$

$$\begin{aligned} \hat{\omega} &\mapsto (\cos^2(\alpha) - \sin^2(\alpha) + 2\sin^2(\alpha))\hat{\omega} = \hat{\omega} \\ \hat{e} &\mapsto \cos(2\alpha)\hat{e} + \sin(2\alpha)\hat{\omega} \times \hat{e} \\ \hat{\omega} \times \hat{e} &\mapsto \cos(2\alpha)\hat{\omega} \times \hat{e} + \sin(2\alpha)\hat{\omega} \times (\hat{\omega} \times \hat{e}) \\ &= \cos(2\alpha)\hat{\omega} \times \hat{e} - \sin(2\alpha)\hat{e} \end{aligned}$$

Rotations

Theorem

The map
$$\tau : \left\{ \begin{array}{ccc} S & \to & \mathrm{SO}(3,\mathbb{R}) \\ s & \mapsto & \tau(s) : \left\{ \begin{array}{ccc} V & \to & V \\ v & \mapsto & svs^{-1} \end{array} \right\} \end{array} \right\}$$
has the properties:

1. $\tau(s)$ *is a specially orthogonal linear transformation of the vector space* V.
2. τ *is an epimorphism with kernel* $\ker \tau = \langle -\mathrm{h}_0 \rangle = \{\mathrm{h}_0, -\mathrm{h}_0\} = S \cap Z(\mathbb{H})$.

Rotations

Theorem

The map $\tau : \left\{ \begin{array}{ccc} S & \to & \mathrm{SO}(3,\mathbb{R}) \\ s & \mapsto & \tau(s) : \left\{ \begin{array}{ccc} V & \to & V \\ v & \mapsto & svs^{-1} \end{array} \right\} \end{array} \right\}$ *has the properties:*

1. $\tau(s)$ *is a specially orthogonal linear transformation of the vector space* V.
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Summary

$$S/_{\{\pm 1\}} \cong \mathrm{SU}(2,\mathbb{C})/_{\{\pm \mathrm{Id}\}} \cong \mathrm{SO}(3,\mathbb{R})$$

Image Recognition

Work Load (Complexity): Number of Multiplications

1. Applying a matrix to a vector: 9 multiplications.
2. Conjugating an imaginary vector by a unit quaternion: 18 multiplications.
3. Multiplication of two matrices: 27 multiplications.
4. Multiplication of two unit quaternions: 16 multiplications.
5. Calculating the rotational matrix of a unit quaternion: 10 multiplications.

From **Wik_Quat**

Image Recognition

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Image Recognition

Task (Determining the Rotation)

Which rotation maps the model $\{\vec{m}_i \mid i = 1, \ldots, n\}$ *to the object in the scenery* $\{\vec{s}_i \mid i = 1, \ldots, n\}$ *?*
A translation may move the object of the scenery so that one point of the model and the image coincide. This point will be chosen to be the origin of the rotation. So we are looking for a rotation D which minimizes the error

$$E(D) = \sum_{i=1}^{n} \|\vec{s}_i - D\vec{m}_i\|^2 \ .$$

Image Recognition

Using Unit Quaternions $q = \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) \\ \sin\left(\frac{\alpha}{2}\right)\hat{\omega} \end{pmatrix}$

$$E(D) = \sum_{i=1}^{n} \|\vec{s}_i - D\vec{m}_i\|^2$$

Image Recognition

Using Unit Quaternions $q = \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) \\ \sin\left(\frac{\alpha}{2}\right)\hat{\omega} \end{pmatrix}$

$$E(D) = \sum_{i=1}^{n} \|\vec{s}_i - D\vec{m}_i\|^2 \cdot 1$$

Image Recognition

Using Unit Quaternions $q = \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) \\ \sin\left(\frac{\alpha}{2}\right)\hat{\omega} \end{pmatrix}$

$$E(D) = \sum_{i=1}^{n} \|\vec{s}_i - D\vec{m}_i\|^2 \cdot 1 = \sum_{i=1}^{n} \|\vec{s}_i - q\vec{m}_i q^{-1}\|^2 \cdot \|q^2\| \quad (1)$$

(1): $\|q^2\| = 1$

Image Recognition

Using Unit Quaternions $q = \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) \\ \sin\left(\frac{\alpha}{2}\right)\hat{\omega} \end{pmatrix}$

$$\begin{aligned} E(D) &= \sum_{i=1}^{n} \|\vec{s}_i - D\vec{m}_i\|^2 \cdot 1 = \sum_{i=1}^{n} \|\vec{s}_i - q\vec{m}_i q^{-1}\|^2 \cdot \|q^2\| \quad (1) \\ &= \sum_{i=1}^{n} \|\vec{s}_i q - q\vec{m}_i\|^2 \end{aligned}$$

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Image Recognition

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$$
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E(D) &= \sum_{i=1}^{n} \|\vec{s}_i - D\vec{m}_i\|^2 \cdot 1 = \sum_{i=1}^{n} \|\vec{s}_i - q\vec{m}_i q^{-1}\|^2 \cdot \|q^2\| && (1) \\
&= \sum_{i=1}^{n} \|\vec{s}_i q - q\vec{m}_i\|^2 = \sum_{i=1}^{n} \|A_i \vec{q}\,\|^2 && (2)
\end{aligned}
$$

(1): $\|q^2\| = 1$

(2): $q \mapsto \vec{s}_i q - q\vec{m}_i$ is $\mathbb{R}$-linear $\mathbb{H} \to \mathbb{H}$ in q: $A_i \in GL(\mathbb{R}^4)$.

Image Recognition

Using Unit Quaternions $q = \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) \\ \sin\left(\frac{\alpha}{2}\right)\hat{\omega} \end{pmatrix}$

$$\begin{aligned} E(D) &= \sum_{i=1}^{n} \|\vec{s}_i - D\vec{m}_i\|^2 \cdot 1 = \sum_{i=1}^{n} \|\vec{s}_i - q\vec{m}_i q^{-1}\|^2 \cdot \|q^2\| \quad (1) \\ &= \sum_{i=1}^{n} \|\vec{s}_i q - q\vec{m}_i\|^2 = \sum_{i=1}^{n} \|A_i\vec{q}\,\|^2 = \sum_{i=1}^{n} \vec{q}^{\,t} A_i^t A_i \vec{q} \quad (2) \end{aligned}$$

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$$
\begin{aligned}
E(D) &= \sum_{i=1}^{n} \|\vec{s}_i - D\vec{m}_i\|^2 \cdot 1 = \sum_{i=1}^{n} \|\vec{s}_i - q\vec{m}_i q^{-1}\|^2 \cdot \|q^2\| \quad (1) \\
&= \sum_{i=1}^{n} \|\vec{s}_i q - q\vec{m}_i\|^2 = \sum_{i=1}^{n} \|A_i\vec{q}\,\|^2 = \sum_{i=1}^{n} \vec{q}^{\,t} A_i^t A_i \vec{q} \quad (2) \\
&= \vec{q}^{\,t} \left(\sum_{i=1}^{n} A_i^t A_i \right) \vec{q}
\end{aligned}
$$

(1): $\|q^2\| = 1$

(2): $q \mapsto \vec{s}_i q - q\vec{m}_i$ is $\mathbb{R}$-linear $\mathbb{H} \to \mathbb{H}$ in q: $A_i \in GL(\mathbb{R}^4)$.

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Using Unit Quaternions $q = \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) \\ \sin\left(\frac{\alpha}{2}\right)\hat{\omega} \end{pmatrix}$

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(1): $\|q^2\| = 1$
(2): $q \mapsto \vec{s}_i q - q\vec{m}_i$ is $\mathbb{R}$-linear $\mathbb{H} \to \mathbb{H}$ in q: $A_i \in \mathrm{GL}(\mathbb{R}^4)$.
(3): B is symmetric and (semi-)definite.

Image Recognition

$$\vec{q}^{\,t} \cdot B \cdot \vec{q} \;=\; \sum_{i=1}^{n} \|A_i \vec{q}\|^2$$

is (semi-)definite. The eigen vector of the smallest non-negative eigen value minimizes the error.

Image Recognition

$$\vec{q}^{\,t} \cdot B \cdot \vec{q} \;=\; \sum_{i=1}^{n} \|A_i \vec{q}\|^2$$

is (semi-)definite. The eigen vector of the smallest non-negative eigen value minimizes the error.

Method

With $A_i : \left\{ \begin{array}{ccc} \mathbb{H} & \rightarrow & \mathbb{H} \\ q & \mapsto & \vec{s}_i q - q\vec{m}_i \end{array} \right\} \in \mathrm{GL}_{\mathbb{R}}(\mathbb{H})$ *and* $B = \sum_{i=1}^{n} A_i^t A_i$ *the unit eigen vector of the smallest eigen value of the matrix* B *minimizes the error* $E(D)$*. The smallest eigen value and its corresponding unit eigen value may be calculated using the von Mises' or Wielandt's algorithm.*

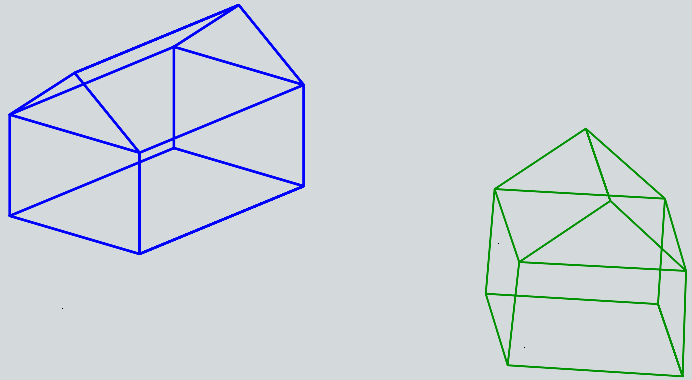
Model and Scenery

Model, Scenery

```
mo←4 3⍴0 0 0 12 0 0 12 8 0 0 8 0
mo←mo,[1]0 0 5+[2]mo
mo←mo,[1]2 3⍴0 4 8 12 4 8
sc←mo+.×1 Drm3 ¯45 4 5
sc←(0.99+(⍴sc)⍴0.02×∊((⍴,sc)⍴1)?¨2)×sc
sc←14 31 4+[2]sc
```

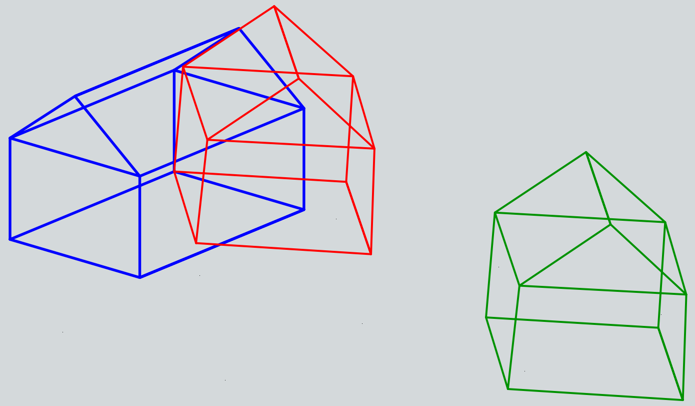
Model and Scenery

Model, Scenery



Model and Scenery

Model, Scenery, Translation of the Object of the Scenery



Model and Scenery

Method

With $A_i : \left\{ \begin{array}{ccc} \mathbb{H} & \rightarrow & \mathbb{H} \\ q & \mapsto & \vec{s}_i q - q\vec{m}_i \end{array} \right\} \in \mathrm{GL}_{\mathbb{R}}(\mathbb{H})$ *and* $B = \sum_{i=1}^{n} A_i^t A_i$ *the unit eigen vector of the smallest eigen value of the matrix* B *minimizes the error* $E(D)$. *The smallest eigenvalue and its corresponding unit eigen value may be calculated using the von Mises' or Wielandt's algorithm.*

Calculation

Model and Scenery

Method

With $A_i : \left\{ \begin{array}{ccc} \mathbb{H} & \to & \mathbb{H} \\ q & \mapsto & \vec{s}_i q - q\vec{m}_i \end{array} \right\} \in \mathrm{GL}_{\mathbb{R}}(\mathbb{H})$ *and* $B = \sum_{i=1}^{n} A_i^t A_i$ *the unit eigen vector of the smallest eigen value of the matrix* B *minimizes the error* $E(D)$. *The smallest eigenvalue and its corresponding unit eigen value may be calculated using the von Mises' or Wielandt's algorithm.*

Calculation

```
      q←⊂[2]4 4⍴5⊃1
      A←⍉¨↑¨⊂[2]((⊂[2]0,sc)∘.Hmul q)-⍉q∘.Hmul⊂[2]0,mo
      ,(w e)←Wiela1 B←↑+/(⍉¨A)+.×¨A
0.4749283006     0.9226059704
                ¯0.02755600419
                ¯0.04835511334
                 0.3817075752
```

Model and Scenery

Calculation von Mises' Algorithm

```
r←Mises1 mat
x←(1▯ρmat)↑1

DO:
x←mat+.×xalt←x
x←x÷(+/x×x)*0.5
→((⌈/|x-xalt)>1E¯8)/DO

r←((mat+.×x)⌹x)(,[1.5]x)
```

Model and Scenery

Calculation Wielandt's Algorithm

```
r←Wiela1 mat
x←(1▯⍴mat)↑1

DO:
x←(xalt←x)⌹mat
x←x÷(+/x×x)*0.5
→((⌈/|x-xalt)>1E¯8)/DO

r←((mat+.×x)⌹x)(,[1.5]x)
```

Model and Scenery

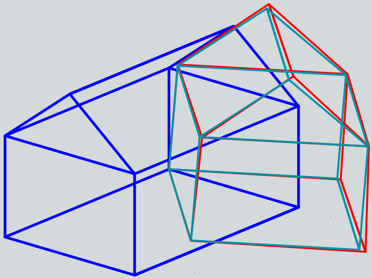
Calculation Wielandt's Algorithm

```
      q←⊂[2]4 4⍴5⊃1
      A←⍉¨↑¨⊂[2]((⊂[2]0,sc)∘.Hmul q)-⍉q∘.Hmul⊂[2]0,mo
      ,(w e)←Wiela1 B←↑+/(⍉¨A)+.×¨A
 0.4749283006    0.9226059704
                ¯0.02755600419
                ¯0.04835511334
                 0.3817075752

      (sc-↑(⊂Hdrmat,e)+.×¨⊂[2]mo)÷sc
1                1               1
0.01195321674    0.02690747054   0.05094570483
0.02025259723    0.009462197457  0.1873015667
0.02697763674    0.01218516893   0.0005827847932
0.0335428643    ¯0.08286290679   0.00986244847
0.01178156164    0.02610348703   0.03224123759
0.04065164481    0.02820517393   0.01477831918
0.008312330371   0.01091595672   0.01137495259
0.009319165317   0.02747665199   0.01028882928
0.03313149529    0.007787749596  0.01245741103
```

Model and Scenery

Recognition



Literature